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Note

A Numerical Test of the Reliability of Galerkin Approximations to the Solutions of the Navier–Stokes Equation

The method of Galerkin (for a general exposition see, for instance, Ref. [1]) is particularly attractive for the approximate solution of the Navier–Stokes equation since in principle the boundary conditions and the energy balance equation are exactly satisfied [2]. Also it is computationally efficient when used in conjunction with appropriate Fast Fourier Transform algorithms [3–5]. In its present state, however, the method suffers from a serious weakness: given an externally applied force $\mathbf{f}(t, \mathbf{r})$ and the value of the viscosity coefficient ν we do not know how to choose the set of Fourier modes in order to guarantee a given accuracy. Worst, we are not even sure that the Galerkin approximation behaves qualitatively like the exact solution with same initial flow, that is, whether both tend to a steady, periodic, quasi-periodic, almost periodic (in the usual mathematical sense), or turbulent state as the time $t \to \infty$.

The purpose of this note is to propose a test which we believe can be used to decide a posteriori if the exact solution behaves like the numerically computed Galerkin approximation. The idea of the test came to us after learning of a remarkable property of the solutions of the Navier–Stokes equation which had been conjectured by Hopf in 1948 [6] and was made precise and rigorously demonstrated by Foias and Prodi in 1967 [7]. Roughly speaking this property is that in a certain sense these solutions become finite-dimensional as $t \to \infty$. Further results on the subject were obtained by Ladyzhenskaya [8, 9] (see also her review [10]) and by Foias and Temam [11, 12]. We shall begin with a brief exposé of the works of these mathematicians which are relevant to our purpose.

Consider the two-dimensional flow of an incompressible viscous fluid with unit density contained in a finite volume Ω with rigid boundary $\partial\Omega$ and governed by the Navier–Stokes equation. The velocity $\mathbf{v}(t, \mathbf{r}) = (v_1(t, \mathbf{r}), v_2(t, \mathbf{r}))$ and the pressure of the fluid are then determined by the equations

$$\partial\mathbf{v}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mathbf{f}(t, \mathbf{r}) + \nu\nabla^2\mathbf{v}, \tag{1.1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.2}$$

the "no-slip" boundary condition

$$\mathbf{v}|_{\partial\Omega} = 0, \tag{1.3}$$

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and some initial condition

$$\mathbf{v}(0,\mathbf{r})=\mathbf{v}_0(\mathbf{r}), \tag{1.4}$$

where $\mathbf{r}=(x_1,x_2)$, p is the pressure, $\mathbf{f}$ the external force, and ν the viscosity coefficient that we assume constant.

It is known that in a variety of appropriate function spaces and for a wide class of forces $\mathbf{f}$ and initial conditions $\mathbf{v}_0$ this problem has a unique and bounded solution defined for all times $t \geqslant 0$ (see, for instance, [10]).

The linear problem

$$\nabla^2 \mathbf{w} + \nabla q = -\lambda \mathbf{w},$$
$$\nabla \cdot \mathbf{w} = 0, \qquad \mathbf{w}|_{\partial\Omega} = 0,$$

is known [10] to uniquely determine both $\mathbf{w}$ and q (up to an additive constant) in Ω and to possess a complete system of orthonormal eigenfunctions $\{\mathbf{w}_i\}$ with positive eigenvalues

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty.$$

Consider a particular solution $\mathbf{v}(t,\mathbf{r})$ of problem (1) and its Fourier representation in terms of the $\mathbf{w}$'s

$$\mathbf{v}(t,\mathbf{r}) = \sum_{i=1}^{\infty} c_i(t)\,\mathbf{w}_i(\mathbf{r}),$$

where the modes c_i are given by the scalar products

$$c_i(t) = \int_\Omega \mathbf{v}(t,\mathbf{r}) \cdot \mathbf{w}_i(\mathbf{r}) \equiv (\mathbf{v}(t,\mathbf{r}), \mathbf{w}_i(\mathbf{r})).$$

Let us break the infinite sum into two parts (from now on we drop the time and space dependence from the formulae) $\mathbf{v}=\mathbf{v}'+\mathbf{v}''$, where

$$\mathbf{v}' = \sum_{i=1}^{n} c_i \mathbf{w}_i \qquad \text{and} \qquad \mathbf{v}'' = \sum_{i=n+1}^{\infty} c_i \mathbf{w}_i,$$

i.e., $\mathbf{v}'$ is the projection $\mathscr{P}_n\mathbf{v}$ of $\mathbf{v}$ on the subspace E_n spanned by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$, and $\mathbf{v}''$ is its projection $\mathscr{Q}_n\mathbf{v}$ on the orthogonal complement of E_n.

Foias and Prodi have shown [7] that if n is so chosen that

$$\lambda_{n+1} > (16/\nu^2)C^2, \tag{2}$$

where C is a certain constant, and if two solutions $\mathbf{v}_1=\mathbf{v}_1'+\mathbf{v}_1''$ and $\mathbf{v}_2=\mathbf{v}_2'+\mathbf{v}_2''$ are such that

$$\lim_{t\to\infty}(\mathbf{v}_1'-\mathbf{v}_2')=0$$

then

$$\lim_{t\to\infty} (\mathbf{v}_1 - \mathbf{v}_2) = 0.$$

This means that if we consider all solutions whose projections $\mathscr{P}_n\mathbf{v} = \mathbf{v}'$ are the same as $t \to \infty$ then they all tend to the same limit as $t \to \infty$. In other words in the limit $t \to \infty$ there is only one solution with projection $\mathbf{v}'$. Hence if we know $\mathbf{v}'$ we can in principle reconstruct $\mathbf{v}$. Suppose instead that we know only $\mathscr{P}_{n'}\mathbf{v}$ with $n' < N$, where N is the smallest n for which inequality (2) holds; then there may be many distinct solutions with the same projection $\mathscr{P}_{n'}\mathbf{v}$ whose higher order modes c_i, $i > n'$, behave in entirely different ways.

Another important result of Foias and Prodi [7] is this: if in the limit $t \to \infty$ the projection $\mathscr{P}_n\mathbf{v} = \mathbf{v}'$, $n \geqslant N$, is stationary, periodic, quasi-periodic, or almost periodic, so is $\mathbf{v}$, respectively. When $\mathbf{v}'$ is stationary the property has been extended to the three-dimensional and time-independent ($\partial\mathbf{v}/\partial t = 0$) Navier–Stokes equation by Foias and Temam [11]. Although not proven yet, it is probable that the behavior of $\mathbf{v}$ is also the same as that of $\mathbf{v}'$ when $\mathbf{v}'$ is chaotic.

These properties imply that if we knew the constant C in inequality (2) and therefore N, and if we could compute the first N modes c_i of a particular solution, we would be guaranteed that the exact solution $\mathbf{v}$ behaves like its projection $\mathscr{P}_N\mathbf{v}$. But again if we know only $\mathscr{P}_{n'}\mathbf{v}$ with $n' < N$ then $\mathbf{v}$ and $\mathscr{P}_{n'}\mathbf{v}$ may behave in completely different ways. In practice we do not know how to calculate the modes c_i. However, it has been shown that the sequence of Galerkin approximations converges strongly to the exact solution with respect to some appropriate norm (see, for instance, [10] and refs therein). We can assume therefore that if $\mathbf{v}^{(k)}$ denotes the Galerkin approximation based on $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ then $c_i^{(k)} = (\mathbf{v}^{(k)}, \mathbf{w}_i) \to c_i$ for all i's as $k \to \infty$. This suggests that for k large enough $\mathbf{v}^{(k)}$ (which is not equal to $\mathscr{P}_k\mathbf{v}$ in general) behaves like the exact solution of the problem

$$\partial\mathbf{u}^{(k)}/\partial t + (\mathbf{u}^{(k)} \cdot \nabla)\mathbf{u}^{(k)} = -\nabla p^{(k)} + \mathscr{P}_k\mathbf{f}(t, \mathbf{r}) + \nu\nabla^2\mathbf{u}^{(k)},$$

$$\nabla \cdot \mathbf{u}^{(k)} = 0, \qquad \mathbf{u}^{(k)}|_{\partial\Omega} = 0, \qquad \mathbf{u}^{(k)}(0, \mathbf{r}) = \mathbf{v}^{(k)}(0, \mathbf{r}).$$

If in addition the projection of the force $\mathbf{f}$ on the orthogonal complement of E_k

$$\mathscr{Q}_k\mathbf{f} = \sum_{i=k+1}^{\infty} (\mathbf{f}, \mathbf{w}_i)$$

is negligible as compared to $\mathscr{P}_k\mathbf{f}$ we can even expect that $\mathbf{v}^{(k)}$ behaves like the exact solution of problem (1) defined by the initial condition $\mathbf{v}_0(\mathbf{r}) = \mathbf{v}^{(k)}(0, \mathbf{r})$. In his recent review of turbulent phenomena [13] Rabinovich has stated that this is indeed the case even when $\mathbf{v}^{(k)}$ has a continuous time-frequency spectrum but there is nothing to support this assertion in the work of Ladyzhenskaya [9] that he quoted in this connection. We have learned, however, that Foias and Temam [12] have just made the following important step regarding this question:

Let us write the ordinary differential equations for the modes in vector form as

$$d\mathbf{c}^{(k)}/dt = \mathbf{G}_k(\mathbf{c}^{(k)}), \tag{3}$$

where $\mathbf{c}^{(k)} = (c_1^{(k)},..., c_k^{(k)})$, and assume that

(a) $k \geqslant N$,

(b) $\mathbf{f}(t, \mathbf{r})$ becomes independent of t as $t \to \infty$, i.e., $\mathbf{f}(t, \mathbf{r}) \to \mathbf{f}_\infty(\mathbf{r})$,

(c) Eq. (3) has a stable stationary solution $\mathbf{c}_\infty^{(k)}$ with basin of attraction B, i.e.,

$$\lim_{t\to\infty} \mathbf{c}^{(k)}(t) = \mathbf{c}_\infty^{(k)} \qquad \text{for all} \quad \mathbf{c}^{(k)}(0) \in B,$$

(d) in some sense $\mathscr{Q}_k \mathbf{f}_\infty$ is negligible as compared to $\mathscr{P}_k \mathbf{f}_\infty$.

Then the exact solution of problem (1) also tends to a stable stationary solution $\mathbf{v}_\infty(\mathbf{r})$ for all initial $\mathbf{v}_0(\mathbf{r}) = \sum_{i=1}^k c_i^{(k)}(0)\, \mathbf{w}_i(\mathbf{r})$ such that $\mathbf{c}^{(k)}(0) \in B$.

This result lends support to the conjecture that in general the Galerkin approximations $\mathbf{v}^{(k)}$ behave like the exact solutions with same initial data provided $k \geqslant N$ and $\mathscr{Q}_k \mathbf{f}(t, \mathbf{r})$ is negligible as compared to $\mathscr{P}_k \mathbf{f}(t, \mathbf{r})$.

The value of N is determined from inequality (2) as soon as we know C^2 which can be taken as any upper bound for the quantity

$$\Gamma^2(\mathbf{v}) = \int_\Omega (\nabla \mathbf{v})^2 = \int_\Omega ((\nabla v_1)^2 + (\nabla v_2)^2)$$

in the limit $t \to \infty$. The best value which can be used for the determination of N is therefore the least upper bound of Γ^2. Because a sharp a priori estimate of this l.u.b. is difficult to obtain, we propose the following numerical test:

Choose a k and evaluate $\Gamma^2(\mathbf{v}^{(k)})$ along as the ordinary differential equations for the modes $c_i^{(k)}$ are integrated. After the initial transient of duration T, say, $\Gamma^2(\mathbf{v}^{(k)})$ will either tend to a constant if the asymptotic state is stationary or oscillate otherwise. In either case an upper bound $\bar{\Gamma}_k^2$ for $\Gamma^2(\mathbf{v}^{(k)})$ can be determined for large times. Suppose that the following conditions are satisfied:

(C1) $\lambda_{k+1} > (16/\nu^2)\, \bar{\Gamma}_k^2$,

(C2) in some sense $\mathscr{Q}_k \mathbf{f}(t, \mathbf{r})$ is negligible as compared to $\mathscr{P}_k \mathbf{f}(t, \mathbf{r})$ for $t > T$.

We can then conclude from the foregoing that the exact solution $\mathbf{v}(t, \mathbf{r})$ with initial flow $\mathbf{v}(0, \mathbf{r}) = \mathbf{v}^{(k)}(0, \mathbf{r})$ certainly behaves like $\mathbf{v}^{(k)}$. But if at least one of these conditions is not satisfied the behaviors of $\mathbf{v}$ and $\mathbf{v}^{(k)}$ may be entirely different and it is necessary to perform the test for increasing values of k until both conditions are satisfied.

A word of caution is in order at this point: we have tacitly assumed that the force $\mathbf{f}(t, \mathbf{r})$ remains the same throughout the sequence of tests which may have to be performed. If instead $\mathbf{f}$ is determined through coupling with other equations as in convection or MHD problems different forces $\mathbf{f}_k$ will be generated for each value of k.

In such cases a successful test for $k \geqslant k_c$, say, only means that the exact solution of the Navier–Stokes equation *driven by the particular force* $\mathbf{f}_k$ behaves like the Galerkin approximation $\mathbf{v}^{(k)}$; it certainly does not mean that the exact solution of the complete system of partial differential equations behaves like its Galerkin approximation, unless there are good reasons to believe that $\mathbf{f}_k$ and the other variables are closely approximated. In all rigor a reliable test for such systems can be devised only if and when the theory presented above is extended to them. We feel nevertheless that the results of performing the test described above on Navier–Stokes equations coupled with other equations may be useful for the extension of the theory.

Finally we want to make the following remarks:

(a) In deriving inequality (2) Foias and Prodi were primarily interested in showing the finite-dimensionality of the solutions for large times. They did not try to get the best possible estimate for the numerical factor multiplying the ratio C^2/ν^2. Thus the value 16 that they indicated is probably much larger than is necessary for their result to hold. C. Foias has told us that he can derive a better estimate for this factor.

(b) The condition (C2) that $\mathscr{Q}_k \mathbf{f}(t, \mathbf{r})$ be much smaller than $\mathscr{P}_k \mathbf{f}(t, \mathbf{r})$ needs to be formalized in a manner consistent with the latest results of Foias and Temam quoted above.

(c) We have tacitly assumed throughout our exposé that in the limit $t \to \infty$ the exact solution of the Navier–Stokes equation is structurally stable in the sense that small changes in the initial flow and in the force do not affect its ultimate characteristics.

We will report on these questions later as further results become available.

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